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A Characterization of Smooth Normed Linear Spaces*

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In this paper a new characterization of smooth normed linear spaces is discussed using the notion of proximal points of a pair of convex sets. It is proved that a normed linear space is smooth if and only if for each pair of convex sets, points which are mutually nearest to each other from the respective sets are proximal.

A normed linear space X is called smooth if each point of the unit sphere $S(X) = \{x \in X / \| x \| = 1\}$ has a unique support hyperplane to the closed unit ball $M(X) = \{x \in X / \| x \| \leqslant 1\}$, or equivalently, if to each $x \in X, x \neq 0$, there corresponds a unique Hahn–Banach functional $L \in X^*$ satisfying $\| L \| = 1$ and $L(x) = \| x \|$.

Some of the well-known approximation theoretic characterizations of smooth normed linear spaces are contained in the following theorem.

THEOREM 1 (cf. *Singer* [8, pp. 112], Phelps [7, pp. 240], *and Cudia* [2, p. 93]). *The following statements for a normed linear space X are equivalent*:

(1) *X is smooth.*

(2) *All $\sigma(X^*, X)$-closed linear subspaces of X^*, of a certain fixed finite codimension m, where $1 \leqslant m \leqslant \dim X^* - 1$, are Chebyshev subspaces.*

(3) *All linear subspaces of X of a certain fixed finite dimension n, where $1 \leqslant n \leqslant \dim X - 1$, have the property (U) of unique Hahn–Banach extension* (cf. *Phelps* [7]).

If X^* is strictly convex, then X is smooth and in addition if X is reflexive then it is well-known that there is a complete duality between smoothness and strict convexity.

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In the present paper we discuss a characterization of smooth norm in terms of proximal points of convex sets. This appears to be new and different from the other known characterizations of smoothness of a norm in approximation theory.

Let U, V be a pair of convex sets in a normed linear space X. We call the points $\bar{u} \in U$, $\bar{v} \in V$ proximal points of the sets U, V if

$$\| \bar{u} - \bar{v} \| = d(U, V) = \inf_{u \in U, v \in V} \| u - v \| .$$

If the points $\bar{u} \in U$, $\bar{v} \in V$ are proximal, then they are clearly the points mutually nearest to each other from the respective set. However, the converse of this statement is in general not true. This is illustrated by the following examples.

EXAMPLE 1. Let $X = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = \max(| x_1 |, | x_2 |)$. Take $U = \{(\alpha, 0)/1 \leqslant \alpha \leqslant 2\}$ and $V = \{(0, \beta)/1 \leqslant \beta \leqslant 2\}$. Here $d(U, V) = 1$ and the proximal points correspond to $\alpha = \beta = 1$. For a fixed α_0, $1 \leqslant \alpha_0 \leqslant 2$, all the points $(0, \beta)$, $\beta \leqslant \alpha_0$ of V are nearest points to the point $(\alpha_0, 0)$ of U and likewise for a fixed β_0, $1 \leqslant \beta_0 \leqslant 2$, all the points $(\alpha, 0)$, $\alpha \leqslant \beta_0$ of U are nearest points to the point $(0, \beta_0)$ of V. Thus for any γ, $1 \leqslant \gamma \leqslant 2$, the points $(\gamma, 0)$, $(0, \gamma)$ of U, V respectively, are points mutually nearest to each other from the other set. However, the points corresponding to $\gamma = 1$ are the only proximal points. In the same example, if we take U, V as the open line segments instead of the closed line segments the proximal points, in fact, do not exist.

EXAMPLE 2. Let $X = C[0, 1]$ with the supremum norm.

Take $U = \{(1 - \alpha) + \alpha t/0 \leqslant \alpha \leqslant 1\}$ and $V = \{\beta t/0 \leqslant \beta \leqslant 1\}$.
Here $d(U, V) = 0$ and the proximal points correspond to $\alpha = \beta = 1$.

$$\begin{aligned} \|(1 - \alpha) + \alpha t - \beta t \| &= 1 - \alpha, \text{ if } \alpha \leqslant \beta \\ &= 1 - \beta, \text{ if } \alpha > \beta. \end{aligned}$$

Thus for any γ, $0 \leqslant \gamma \leqslant 1$, the points $(1 - \gamma) + \gamma t$ of U and γt of V are points mutually nearest to each other from the other set; but the points corresponding to $\gamma = 1$ are the only proximal points.

In the preceeding examples the sets U, V are non-Chebyshev. The following example illustrates that even for Chebyshev sets U, V, points which are mutually nearest to each other need not be proximal.

EXAMPLE 3. Let $X = \mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\| = | x_1 - x_2 | + (x_1^2 + x_2^2)^{1/2}.$$

The norm $\|\cdot\|$ is strictly convex, being the sum of a seminorm and a strictly convex norm. Let $U = \{(x_1, 0) | -1 \leqslant x_1 \leqslant 1\}$ and $V = \{(0, x_2) | -1 \leqslant x_2 \leqslant 1\}$. Then the sets U, V are Chebyshev for which the points $(\alpha, 0)$, $(0, -\alpha)$, $-1 \leqslant \alpha \leqslant 1$ are mutually nearest to each other from the respective set. However, $(0, 0)$, $(0, 0)$ are the only proximal points.

Motivated by these examples, we introduce the following property (P) for a normed linear space X.

(P) For each pair U, V of convex sets in X and points $\bar{u} \in U$, $\bar{v} \in V$, $\bar{u}$ being a nearest point of $\bar{v}$ in U and $\bar{v}$ being a nearest point of $\bar{u}$ in V, imply that $\bar{u}$, $\bar{v}$ are proximal points of U, V.

In [1], Cheney and Goldstein proved that (P) holds for a Hilbert space X when U, V are closed convex sets. In [5], we proved that if X is a normed space whose dual X^* is strictly convex, then (P) holds for X. Here we extend this result to prove that X is smooth if and only if (P) holds for X. For this purpose, the following characterization theorem for proximal points is employed.

THEOREM 2 (cf. [5]). *Let U, V be convex sets. Then $\bar{u} \in U$, $\bar{v} \in V$ are proximal if and only if there exists an $L \in X^*$ such that*

(1) $\|L\| = 1$,

(2) $\operatorname{Re} L(u - \bar{u}) \geqslant 0$ for each $u \in U$,

(3) $\operatorname{Re} L(v - \bar{v}) \leqslant 0$ *for each* $v \in V$,

(4) $L(\bar{u} - \bar{v}) = \|\bar{u} - \bar{v}\|$.

The main result of this paper reads as follows.

THEOREM 3. *For a normed linear space X the following statements are equivalent*:

(1) *X is smooth,*

(2) *X satisfies property (P),*

(3) *The norm $\|\cdot\|$ is Gâteaux-differentiable at each nonzero point of X.*

Proof. The equivalence of (1) and (3) is a well-known result of Ascoli–Mazur (cf. [3, pp. 447]). In order to prove (1) $\Rightarrow$ (2), let X be smooth and let U, V be convex sets such that the points $\bar{u} \in U$, $\bar{v} \in V$ are mutually nearest to each other. Then there exist $L_1, L_2 \in X^*$ such that $\|L_1\| = \|L_2\| = 1$,

$$\operatorname{Re} L_1(\bar{u}) = \inf_{u \in U} \operatorname{Re} L_1(u), \qquad \operatorname{Re} L_2(\bar{v}) = \sup_{v \in V} \operatorname{Re} L_2(v)$$

and

$$L_1(\bar{u} - \bar{v}) = L_2(\bar{u} - \bar{v}) = \| \bar{u} - \bar{v} \| .$$

Since X is smooth, one has $L_1 = L_2$ and employing the sufficiency part of Theorem 2 it follows that $\bar{u} \in U$, $\bar{v} \in V$ are proximal.

It remains to show that (2) $\Rightarrow$ (1). Suppose X is not smooth. Then there exist $\bar{x} \in X$, $\bar{x} \neq 0$ and $L_1, L_2 \in S(X^*)$, $L_1 \neq L_2$, such that

$$L_1(\bar{x}) = L_2(\bar{x}) = \| \bar{x} \| .$$

ASSERTION. *There exists an element* $\tilde{x} \in X$ *such that* $0 < \operatorname{Re} L_1(\tilde{x}) < \operatorname{Re} L_2(\tilde{x})$.

Suppose the contrary. Then the half-spaces $\{x \in X / \operatorname{Re} L_1(x) > 0\}$, $\{x \in X / \operatorname{Re}(L_2 - L_1)(x) > 0\}$ have a void intersection. This implies that

$$\varphi(X) \cap \mathbb{R}_+^2 = \varnothing, \qquad \text{where } \varphi : x \in X \to (\operatorname{Re} L_1(x), \operatorname{Re}(L_2 - L_1)(x)) \in \mathbb{R}^2$$

and

$$\mathbb{R}_+^2 = \{(\xi, \eta) / \xi > 0, \eta > 0\}.$$

By the separation form of the Hahn–Banach theorem, there exists a hyperplane $H = \{(\xi, \eta) / \alpha\xi + \beta\eta = \gamma\}$ in $\mathbb{R}^2$ such that $\varphi(X) \subset H$ and $\varphi(X) \cap \mathbb{R}_+^2 = \varnothing$. Hence,

$$\alpha \operatorname{Re} L_1(x) + \beta \operatorname{Re}(L_2 - L_1)(x) = \gamma \qquad (x \in X)$$

and

$$\alpha\xi + \beta\eta > \gamma \text{ (without loss of generality)} \qquad (\xi, \eta) \in \mathbb{R}_+^2.$$

These two together imply that $\alpha \geqslant 0$, $\beta \geqslant 0$ (not both zero simultaneously) and $\alpha \operatorname{Re} L_1 + \beta \operatorname{Re}(L_2 - L_1) = 0$, which contradicts the choice of L_1 and L_2 and establishes the assertion. Now let

$$\hat{x} = \frac{\| \bar{x} \| \, \tilde{x}}{\operatorname{Re} L_2(\tilde{x})} .$$

Then

$$\operatorname{Re} L_1(\hat{x}) < \| \bar{x} \| = \operatorname{Re} L_2(\hat{x}).$$

Take

$$U = \{u \in X / \operatorname{Re} L_2(u) \geqslant L_2(\bar{x}) = \| \bar{x} \|\}$$

and

$$V = \{v \in V / \operatorname{Re} L_1(v) = 0\}.$$

Then 0 is a nearest point to $\bar{x}$ in V and $\bar{x}$ is a nearest point to 0 in U. However, $\bar{x}$ and 0 are not proximal points of U, V. In fact,

$$\hat{x} \in U,\ \hat{x} - \operatorname{Re} L_1(\hat{x})(\bar{x}/\|\bar{x}\|) \in V$$

and

$$\|\hat{x} - (\hat{x} - \operatorname{Re} L_1(\hat{x})(\bar{x}/\|\bar{x}\|))\| = \operatorname{Re} L_1(\hat{x}) < \|\bar{x}\|.$$

This establishes the theorem.

Theorem 3 remains valid, with norm replaced by a seminorm. In fact, it extends to a slightly more general situation. Let X be *a* Hausdorff locally convex linear topological space and let X^* be its topological dual. If f is a convex functional defined on X, then the subdifferential of f at $\bar{x}$ is defined as

$$\partial f(\bar{x}) = \{L \in X^*/f(x) \geqslant f(\bar{x}) + \operatorname{Re} L(x - \bar{x}),\ \forall x \in X\}.$$

It is well known that if f is finite and continuous at $\bar{x}$, then $\partial f(\bar{x}) \neq \varnothing$ and that $\partial f(\bar{x})$ consists of a single element if and only if f is Gâteaux differentiable at $\bar{x}$. Also, f is minimized on a convex set K at $\bar{x}$ if and only if there exists an $L \in \partial f(\bar{x})$ such that $\operatorname{Re} L(x - \bar{x}) \geqslant 0$ for each $x \in K$ (cf. [4, Chap. VIII]). Now suppose that f is a continuous sub-linear functional defined on X, i.e., f is a continuous real-valued function on X satisfying $f(x_1 + x_2) \leqslant f(x_1) + f(x_2)$ for all $x_1, x_2 \in X$ and $f(\lambda x) = \lambda f(x)$ for $x \in X$ and $\lambda \geqslant 0$. Then the subdifferential of f at $\bar{x}$ is given by $\partial f(\bar{x}) = \{L \in \partial f(\theta)/f(\bar{x}) = \operatorname{Re} L(\bar{x})\}$, where θ denotes the zero vector in X. Using this expression for the subdifferential the proof of Theorem 3 could be easily adapted to establish the following theorem.

THEOREM 4. *Let X be a Hausdorff locally convex linear topological space and let f be a continuous sublinear functional defined on X. Then in order that for arbitrarily given convex sets U, V in X and the points $\bar{u} \in U$, $\bar{v} \in V$, the two optimality relations*

$$\varnothing \neq f(\bar{u} - \bar{v}) = \inf_{u \in U} f(u - \bar{v}) = \inf_{v \in V} f(\bar{u} - v)$$

imply the optimality relation

$$f(\bar{u} - \bar{v}) = \inf_{u \in U, v \in V} f(u - v),$$

it is necessary and sufficient that f be Gâteaux differentiable at each point $x \in X$ where $f(x) \neq 0$.

Note added in proof. The author learned recently that the implication (1) ⇒ (2) of Theorem 3 was also given in the research report, Distance between two convex sets, G. Bradley and L. Willner, Report No. 14, Admin. Sciences, Yale University (1969).

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